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Actions of Some Noetherian Hopf Algebras on Path Algebras

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Outline

Acknowledgements

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Motivating Example

The classical notion of symmetry can be encoded using the language of group actions.

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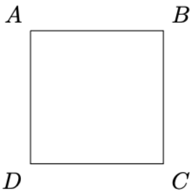

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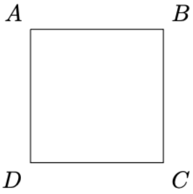

D_4 acts on the vertices of a square $S = \{A, B, C, D\}$ by permuting the vertices.

Motivation

The structure of a Hopf algebra encodes a generalized notion of symmetry, including quantum symmetries.

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Goal: Study how certain Hopf algebras act on path algebras of quivers.

Definition (Algebra)

A $\mathbb{k}$-**algebra** is a ring $(A, +, \cdot)$ with unity 1 such that A also has a $\mathbb{k}$-vector space structure such that for all $\lambda \in \mathbb{k}$ and all $a, b \in A$, we have

$$\lambda(ab) = (\lambda a)b = a(\lambda b) = (ab)\lambda.$$

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Examples:

- The group algebra,

$$\Bbbk G = \{\sum_{g \in G} a_g g | a_g = 0 \text{ for all but finitely many } g \in G\}.$$

Multiplication is given by $(\sum_{g \in G} a_g g)(\sum_{g \in G} b_g g) = \sum_{g,h \in G}(a_g b_g)gh$.

Definition (Coalgebra)

A **coalgebra** over a field $\Bbbk$ is a vector space C over $\Bbbk$ together with $\Bbbk$-linear maps $\Delta : C \to C \otimes C$ and $\varepsilon : C \to \Bbbk$ such that

1. $(id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta$
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Examples:

- The group algebra of G, $\Bbbk[G]$.
  For every $g \in G$, $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$.

Definition (Hopf Algebra)

A **Hopf algebra**, H, has compatible algebra and coalgebra structures denoted $(H, \mu, \eta, \Delta, \varepsilon)$ and is equipped with an antipode map, $S : H \to H$ such that $m \circ (id \otimes S)\Delta = m \circ (S \otimes id)\Delta = \eta \circ \varepsilon$.

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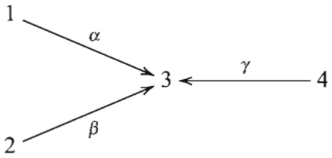
Fact: In every Hopf algebra, the elements where $\Delta(g) = g \otimes g$ form a group called the **grouplike elements of** H. We often denote this group by $G(H)$.

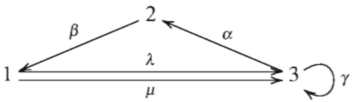
We now turn our attention to quivers.

Definition (Quiver)

A **quiver**, $Q = (Q_0, Q_1, s, t)$ consists of a set of vertices, Q_0, a set of arrows, Q_1, a map sending an arrow to its starting point $s : Q_1 \longrightarrow Q_0$, and a map sending an arrow to its terminal point $t : Q_1 \longrightarrow Q_0$.

Examples:





Definition (Path Algebra, [Sch14])

Let Q be a quiver. The **path algebra** $\Bbbk Q$ of Q is the algebra with basis the set of all paths in the quiver Q and with multiplication defined on two basis elements p, p' by

$$pp' = \begin{cases} p \cdot p' & \text{if } s(p') = t(p) \\ 0 & \text{otherwise.} \end{cases}$$

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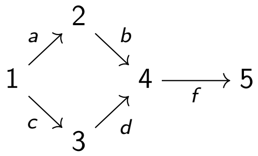
Example: If $p = ab$ and $p' = f$, then $p \cdot p' = abf$.



Background

Goal: Study how certain Hopf algebras act on path algebras of quivers.

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Because every path can be built using vertices and arrows (by concatenation), we only need consider how these Hopf algebras act on Q_0 and Q_1.

Hopf-Ore Extensions

We now turn our attention to Hopf-Ore extensions as a more specific class of Hopf algebras.

By a classification of Panov, every Hopf-Ore extension of $\Bbbk G$ is of the following form.

Let $\chi : G \to \Bbbk$ be a group character and $\alpha : G \to \Bbbk$ be such that $\alpha(uv) = \alpha(u) + \chi(u)\alpha(v)$.

Definition ([Pan03])

Let $A = \Bbbk G$ for some group G and $R = \Bbbk G(\chi, h, \delta)$ be a Hopf algebra over $\Bbbk$. The Hopf algebra $R = \Bbbk G(\chi, h, \delta)$ is a **Hopf-Ore extension** if

1. R is the $\Bbbk$-algebra generated by the variable x and the algebra $\Bbbk G$ and subject to the relation $xg = \chi(g)gx + \alpha(g)(1 - h)g$ for all $g \in G$
2. $\Bbbk G$ is a Hopf subalgebra of R and
3. $\Delta(x) = x \otimes h + 1 \otimes x$ for some $h \in Z(G)$.

Background Information

Definition

Let H be a Hopf algebra and an algebra A.
A **(left) Hopf action** of H on A consists of a left H-module structure on A satisfying:

1. $h \cdot (pq) = \sum_i (h_{i,1} \cdot p)(h_{i,2} \cdot q)$ for all $h \in H$ and $p, q \in A$ where $\Delta(h) = \sum_i h_{i,1} \otimes h_{i,2}$ and
2. $h \cdot 1_A = \varepsilon(h) 1_A$ for all $h \in H$.

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Example: Let $e_1, e_2 \in \Bbbk Q_0$ and $g \in G(H)$. We know $\Delta(g) = g \otimes g$. Then

$$g \cdot (e_1 e_2) = (g \cdot e_1)(g \cdot e_2).$$

Actions of Hopf Ore Extensions of Group Algebras

Let $R = \Bbbk G(\chi, h, \delta)$ be a Hopf-Ore extension of the group algebra, $\Bbbk G$, as described earlier. Let Q be a quiver with path algebra $\Bbbk Q$ and vertex set Q_0.

Proposition

1. *The following data determines a Hopf action of R on $\Bbbk Q_0$.*
   - **1.1** *A permutation action of G on the set Q_0;*
   - **1.2** *A collection of scalars $(\gamma_i \in \Bbbk)_{i \in Q_0}$ such that*

$$\gamma_{g \cdot i} = \chi(g)\gamma_i + \alpha(g) \text{ for all } i \in Q_0 \text{ and for all } g \in G.$$

The x-action is given by

$$x \cdot e_i = \gamma_i e_i - (\chi(h)\gamma_i + \alpha(h))e_{h \cdot i} \qquad \text{for all } i \in Q_0.$$

2. *Every action of R on $\Bbbk Q_0$ is of the form above.*

Proposition

1. *The following data determines a Hopf action of $R = \Bbbk G(\chi, h, \delta)$ on $\Bbbk Q$.*
   - **1.1** *A Hopf action of R on $\Bbbk Q_0$;*
   - **1.2** *A representation of G on $\Bbbk Q_1$ satisfying $s(g \cdot a) = g \cdot sa$ and $t(g \cdot a) = g \cdot ta$ for all $a \in Q_1$ and all $g \in G$;*
   - **1.3** *A $\Bbbk$-linear endomorphism $\sigma : \Bbbk Q_0 \oplus \Bbbk Q_1 \longrightarrow \Bbbk Q_0 \oplus \Bbbk Q_1$ satisfying some technical conditions. With this data, the x-action on $a \in Q_1$ is given by*

$$x \cdot a = \gamma_{ta} a - (\chi(h)\gamma_{sa} + \alpha(h))(h \cdot a) + \sigma(a).$$

2. *Every (filtered) action of R on $\Bbbk Q$ is of the form above.*

Consider the Hopf algebra $H(n, t, q)$.

Definition ($H(n, t, q)$, [LWZ07])

Let n, m, t be integers and q an nth primitive root of unity. $H(n, t, q)$ is defined as the k-algebra generated by x and g subject to the relations

$$g^n = 1 \text{ and } xg = q^m gx.$$

The coalgebra structure on $H(n, t, q)$ is defined by

$$\Delta(g) = g \otimes g \text{ and } \Delta(x) = x \otimes 1 + g^t \otimes x.$$

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In short, the action of $H(n,t,q)$ on $\Bbbk Q$ is given by a collection of scalars, $(\gamma_i \in \Bbbk)_{i \in Q_0}$ such that for all $e_i \in Q_0$ and $a \in Q_1$,

$$x \cdot e_i = \gamma_i e_i - q^{mt} \gamma_i e_{g^{-t} \cdot i} \text{ and } x \cdot a = \gamma_{ta} a - q^m \gamma_{sa} (g^{-t} \cdot a) + \sigma(a).$$

Consider the Hopf algebra $C(n, q)$.

Definition ($C(n, q)$, [Goo89])

Given $n \in \mathbb{Z}$ and $q \in \Bbbk^\times$, let $C(n, q)$ be the $\Bbbk$-algebra given by generators $g^{\pm 1}$ and x subject to the relation

$$xg = q^r gx + g^n - g.$$

The unique Hopf algebra structure on $C(n, q)$ is given by

$$\Delta(g) = g \otimes g \text{ and } \Delta(x) = x \otimes g^{n-1} + 1 \otimes x.$$

Applications to Specific Hopf Algebras

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$$\chi(g) = q^r, h = g^{n-1}, \text{ and } \alpha(g) = -1 \text{ so } \alpha(g)(1 - g^{n-1})g = g^n - g$$

The unique Hopf algebra structure on $C(n,q)$ is given by

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IOWA

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The unique Hopf algebra structure on $C(n,q)$ is given by

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If q is an nth root of unity or an rth root of unity,

$$x \cdot e_i = \gamma_i e_i - q^{r(n-1)}\gamma_i e_{g^{n-1}.i} \text{ and } \quad x \cdot a = \gamma_{ta} a - q^{r(n-1)}\gamma_{sa}(g^{n-1} \cdot a) + \sigma(a)$$

Future Directions

- Continue to find specific Hopf algebras where we can apply our general results.
- Classify the actions of a more general Hopf-Ore extension.
  - $\Delta(x) = x \otimes a + b \otimes x + v(x \otimes x) + w$ for $a, b \in R$ and $v, w \in R \otimes R$

Thank you!

Are there any questions?

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Action of skew-primitives

In the same setting as before with $x \in H$ such that $\Delta(x) = x \otimes h + 1 \otimes x$

Proposition

Every (filtered) action of $x \in H$ on $\Bbbk Q$ is given by

1. *A Hopf action of $x \in H$ on $\Bbbk Q_0$*
2. *A representation of $G(H)$ on $\Bbbk Q_1$ satisfying $s(g \cdot a) = g \cdot sa$ and $t(g \cdot a) = g \cdot ta$ for all $a \in Q_1$ and all $g \in G$.*
3. *A $\Bbbk$-linear endomorphism $\sigma : \Bbbk Q_0 \oplus \Bbbk Q_1 \to \Bbbk Q_0 \oplus \Bbbk Q_1$ satisfying*
   3.1 $\sigma(\Bbbk Q_0) = 0$
   3.2 $\sigma(a) = e_{sa}\sigma(a)e_{h \cdot ta}$ *for all $a \in Q_1$*

   With this data, the x-action on $a \in Q_1$ is given by

$$x \cdot a = \gamma_{ta} a - \gamma_{h \cdot sa}(h \cdot a) + \sigma(a).$$

Proposition

1. *The following data determines a Hopf action of $R = \Bbbk G(\chi, h, \delta)$ on $\Bbbk Q$.*
   1.1 *A Hopf action of R on $\Bbbk Q_0$;*
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   1.3 *A $\Bbbk$-linear endomorphism $\sigma : \Bbbk Q_0 \oplus \Bbbk Q_1 \longrightarrow \Bbbk Q_0 \oplus \Bbbk Q_1$ satisfying*
      1.3.1 $\sigma(\Bbbk Q_0) = 0$;
      1.3.2 $\sigma(a) = e_{sa}\sigma(a)e_{h\cdot ta}$ *for all* $a \in Q_1$;
      1.3.3 $\sigma(g \cdot a) = \chi(g)g\sigma(a) + e_{g\cdot sa}\alpha(g)(1 - h)(g \cdot a)e_{gh\cdot ta}$ *for all* $a \in Q_1$ *and* $g \in G$.

   With this data, the x-action on $a \in Q_1$ is given by

$$x \cdot a = \gamma_{ta}a - (\chi(h)\gamma_{sa} + \alpha(h))(h \cdot a) + \sigma(a).$$

2. *Every (filtered) action of R on $\Bbbk Q$ is of the form above.*

Consider $\diamondsuit$ $yK = Ky$ and $yx = xy + x$ with $\beta = 1$.

Proposition

For $q \neq \pm 1$, the action of y on e_i for all $i \in Q_0$ is given by

$$y \cdot e_i = \gamma_i e_i - \gamma_i e_{K^{-1} \cdot i}$$

such that

1. *If $K^{-2} \cdot i = K^{-1} \cdot i = i$, then $\gamma_i = \dfrac{1}{q^2 - 1}$ for all $i \in Q_0$.*
2. *If $K^{-2} \cdot i = i \neq K^{-1} \cdot i$, then $\gamma_i = -\dfrac{1}{2}$ for all $i \in Q_0$. In particular, q is a second root of unity.*
3. *If $K^{-1} \cdot i, K^{-1} \cdot i$, and i are pairwise not equal, $y \cdot e_i = 0$ for all $i \in Q_0$.*