

Elise Askelsen

Actions of Hopf-Ore Extensions on Path Algebras

QuaSy-Con II

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Acknowledgements

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Motivating Example

The classical notion of symmetry can be encoded using the language of group actions.

The structure of a Hopf algebra encodes a generalized notion of symmetry, including quantum symmetries.

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Motivating Example

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The classical notion of symmetry can be encoded using the language of group actions.

The structure of a Hopf algebra encodes a generalized notion of symmetry, including quantum symmetries.

Goal: Study how certain Hopf-Ore extensions act on path algebras of quivers.

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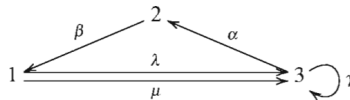
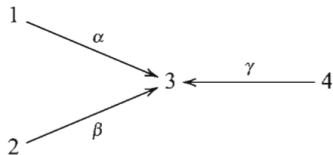
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We first recall some background material about quivers.

Definition (Quiver)

A **quiver**, $Q = (Q_0, Q_1, s, t)$ consists of a set of vertices, Q_0 , a set of arrows, Q_1 , a map mapping an arrow to its starting point $s : Q_1 \rightarrow Q_0$, and a map mapping an arrow to its terminal point $t : Q_1 \rightarrow Q_0$.

Examples:



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Definition (Path Algebra, [Sch14])

Let Q be a quiver. The **path algebra** $\mathbb{k}Q$ of Q is the algebra with basis the set of all paths in the quiver Q and with multiplication defined on two basis elements p, p' by

$$pp' = \begin{cases} p \cdot p' & \text{if } s(p') = t(p) \\ 0 & \text{otherwise.} \end{cases}$$

Definition (Path Algebra, [Sch14])

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$$pp' = \begin{cases} p \cdot p' & \text{if } s(p') = t(p) \\ 0 & \text{otherwise.} \end{cases}$$

- For all $i \in Q_0$, e_i denotes the path of length 0 at i in $\mathbb{k}Q$.
- Moreover, we have the relations $e_i e_j = \delta_{i,j} e_i$ for all $i \in Q_0$ and $e_{sa} a = a = a e_{ta}$ for all $a \in Q_1$.

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To classify actions of Hopf algebras on path algebras, we use properties of $\mathbb{k}Q$.

Nice Properties of $\mathbb{k}Q$:

- $\mathbb{k}Q$ is graded, $\mathbb{k}Q = \bigoplus_{l=0}^{\infty} \mathbb{k}Q_l$.
- $\mathbb{k}Q \cong T_{\mathbb{k}Q_0}(\mathbb{k}Q_1)$
 $= \mathbb{k}Q_0 \oplus \mathbb{k}Q_1 \oplus (\mathbb{k}Q_1 \otimes \mathbb{k}Q_1) \oplus \dots$

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Properties of $\mathbb{k}Q$

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Hence we only need consider how these Hopf algebras act on Q_0 and Q_1 .

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Goal: Extend work of Kinser and Oswald classifying actions of $U_q(\mathfrak{b})$ on $\mathbb{k}Q$.

Motivation for Hopf-Ore Extensions

Goal: Extend work of Kinser and Oswald classifying actions of $U_q(\mathfrak{b})$ on $\mathbb{k}Q$.

We first considered Noetherian Hopf algebras of GK dimension 1 and 2.

Algebra	Generators	Relations	Comultiplication
$U_q(\mathfrak{b})$	$x, g^{\pm 1}$	$gx = qxg$	$\Delta(x) = x \otimes g + 1 \otimes x$
$H(n, t, q)$	$x, g^{\pm 1}$	$gx = q^m xg$	$\Delta(x) = x \otimes g^t + 1 \otimes x$
$B(n, w, q)$	$x, g^{\pm 1}, h^{\pm 1}$	$gx = xg, xh = qhx, x^n = 1 - g^w = 1 - h^w$	$\Delta(x) = x \otimes h + 1 \otimes x$
$C(n, q)$	$x, g^{\pm 1}$	$xg = qgx + g^n - g$	$\Delta(x) = x \otimes g^{n-1} + 1 \otimes x$

Table: Various such Hopf Algebras [Goo12]

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Question: Is there a type of Hopf algebra that encompasses these examples?

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Ore Extensions

Let A be a $\mathbb{k}$ -algebra.

Definition

Let τ be an endomorphism of A . A τ -**derivation** of A is any additive map $\delta : A \rightarrow A$ such that $\delta(ab) = \tau(a)\delta(b) + \delta(a)b$ for all $a, b \in A$ and $\delta(\mathbb{k}) = 0$.

The **Ore extension**, $R = A[x; \tau, \delta]$, of the $\mathbb{k}$ -algebra A is the $\mathbb{k}$ -algebra R generated by the variable x and the algebra A with the relation that for all $a \in A$

$$xa = \tau(a)x + \delta(a).$$

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Question: Assuming A is a Hopf algebra, when is R a Hopf algebra?

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Definition

Let A and $R = A[x; \tau, \delta]$ be Hopf algebras over $\mathbb{k}$. The Hopf algebra $R = A[x; \tau, \delta]$ is a **Hopf-Ore extension** if $\Delta(x) = x \otimes h + 1 \otimes x$ for some $h \in A$ and A is a Hopf subalgebra of R .

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By a classification of Panov in [Pan03],

- There is a character $\chi : A \rightarrow \mathbb{k}$ such that for any $a \in A$

$$\tau(a) = \chi(a_1)a_2$$

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- If A is cocommutative, for $\alpha : A \rightarrow \mathbb{k}$ with $\alpha(uv) = \alpha(u)\varepsilon(v) + \chi(u)\alpha(v)$,

$$\delta(a) := \alpha(a_1)(1 - h)a_2.$$

Motivation for Hopf-Ore Extensions

Goal: Extend work of Kinser and Oswald classifying actions of $U_q(\mathfrak{b})$ on $\mathbb{k}Q$.

We first considered Noetherian Hopf algebras of GK dimension 1 and 2.

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Let $A = \mathbb{k}G$ for some group G and consider the Hopf-Ore Extension,
 $R = A[x; \tau, \delta]$. For some $h \in Z(\mathbb{k}G)$ and for all $g \in G$,

$$\Delta(x) = x \otimes h + 1 \otimes x \quad \text{and} \quad xg = \tau(g)x + \delta(g).$$

Hopf-Ore Extensions of Group Algebras

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By Panov's classification in [Pan03],

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- There is a character $\chi : A \rightarrow \mathbb{k}$ such that for any $a \in A$

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Proposition ([Pan03])

Every Hopf-Ore extension of $\mathbb{k}G$ is of the form above.

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Hopf Action

Goal: Study how Hopf-Ore extensions act on path algebras of quivers.

Definition

Let H be a Hopf algebra and an algebra A .

A **(left) Hopf action** of H on A consists of a left H -module structure on A satisfying:

1. $h \cdot (pq) = \sum_i (h_{i,1} \cdot p)(h_{i,2} \cdot q)$ for all $h \in H$ and $p, q \in A$ where $\Delta(h) = \sum_i h_{i,1} \otimes h_{i,2}$ and
2. $h \cdot 1_A = \varepsilon(h)1_A$ for all $h \in H$.

Example: Let $e_i, e_j \in Q_0$ and $x \in H$ such that $\Delta(x) = x \otimes h + 1 \otimes x$ for $h \in G(H)$. Then

$$x \cdot (e_i e_j) = (x \cdot e_i)(h \cdot e_j) + e_i(x \cdot e_j).$$

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Let $R = \mathbb{k}G(\chi, h, \delta)$ be a Hopf-Ore extension of the group algebra, $\mathbb{k}G$, as described. Let Q be a quiver with path algebra $\mathbb{k}Q$ and vertex set Q_0 .

Theorem (joint work with Kinser)

1. *The following data determines a Hopf action of R on $\mathbb{k}Q_0$.*

1.1 *A permutation action of G on the set Q_0 ;*

1.2 *A collection of scalars $(\gamma_i \in \mathbb{k})_{i \in Q_0}$ such that*

$$\gamma_{g \cdot i} = \chi(g)\gamma_i + \alpha(g) \text{ for all } i \in Q_0 \text{ and for all } g \in G.$$

The x -action is given by

$$x \cdot e_i = \gamma_i e_i - (\chi(h)\gamma_i + \alpha(h))e_{h \cdot i} \quad \text{for all } i \in Q_0.$$

2. *Every action of R on $\mathbb{k}Q_0$ is of the form above.*

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Theorem (joint work with Kinser)

1. *The following data determines a Hopf action of $R = \mathbb{k}G(\chi, h, \delta)$ on $\mathbb{k}Q$.*
 - 1.1 *A Hopf action of R on $\mathbb{k}Q_0$;*
 - 1.2 *A representation of G on $\mathbb{k}Q_1$ satisfying $s(g \cdot a) = g \cdot sa$ and $t(g \cdot a) = g \cdot ta$ for all $a \in Q_1$ and all $g \in G$;*
 - 1.3 *A $\mathbb{k}$ -linear endomorphism $\sigma : \mathbb{k}Q_0 \oplus \mathbb{k}Q_1 \longrightarrow \mathbb{k}Q_0 \oplus \mathbb{k}Q_1$ satisfying some technical conditions. With this data, the x -action on $a \in Q_1$ is given by*

$$x \cdot a = \gamma_{ta}a - (\chi(h)\gamma_{sa} + \alpha(h))(h \cdot a) + \sigma(a).$$

2. *Every (filtered) action of R on $\mathbb{k}Q$ is of the form above.*

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This work allows us to recover the action of $U_q(\mathfrak{b})$ on $\mathbb{k}Q$ as described in Proposition 3.1 and Theorem 3.13 of [KO21].

Next, we consider a specific Noetherian Hopf algebra of GK dimension 2;

Applications to Specific Hopf Algebras

Consider the Hopf algebra $C(n, q)$.

Definition ($C(n, q)$, [Goo89])

Given $n \in \mathbb{Z}$ and $q \in \mathbb{k}^\times$, let $C(n, q)$ be the $\mathbb{k}$ -algebra given by generators $g^{\pm 1}$ and x subject to the relation

$$xg = q^r gx + g^n - g.$$

The unique Hopf algebra structure on $C(n, q)$ is given by

$$\Delta(g) = g \otimes g \text{ and } \Delta(x) = x \otimes g^{n-1} + 1 \otimes x.$$

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Applications to Specific Hopf Algebras

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Definition ($C(n, q)$, [Goo89])

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$$xg = q^r gx + g^n - g.$$

$$\chi(g) = q^r, h = g^{n-1}, \text{ and } \alpha(g)(1 - g^{n-1})g = g^n - g \text{ so } \alpha(g) = -1$$

The unique Hopf algebra structure on $C(n, q)$ is given by

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Theorem (joint work with Kinser)

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1.2 *A collection of scalars $(\gamma_i \in \mathbb{k})_{i \in Q_0}$ such that*

$$\gamma_{g \cdot i} = q^r \gamma_i - 1 \text{ for all } i \in Q_0.$$

The x -action is given by

$$x \cdot e_i = \gamma_i e_i + (1 + q^r + q^{2r} + \cdots + q^{r(n-2)}) \gamma_i e_{g^{n-1} \cdot i} \quad \text{for all } i \in Q_0.$$

2. *Every action of R on $\mathbb{k}Q_0$ is of the form above.*

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$$x \cdot a = \gamma_{ta}a + (1 + q^r + q^{2r} + \cdots + q^{r(n-2)})\gamma_{sa}(g^{n-1} \cdot a) + \sigma(a).$$

2. *Every (filtered) action of $C(n, q)$ on $\mathbb{k}Q$ is of the form above.*

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- Classify the actions of more general Hopf-Ore extensions, $R = A[x; \tau, \delta]$ where the comultiplication is of the form:

$$\Delta(x) = x \otimes a + b \otimes x + v(x \otimes x) + w$$

for $a, b \in A$ and $v, w \in A \otimes A$.

- Classify the indecomposable $\mathbb{k}Q_0$ -bimodules in $\text{rep}(\mathbb{k}G(\chi, h, \delta))$.

Thank you!

Are there any questions?

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Action of skew-primitives

Let H be a Hopf algebra and $x \in H$ such that $\Delta(x) = x \otimes h + 1 \otimes x$

Proposition

Every (filtered) action of $x \in H$ on $\mathbb{k}Q$ is given by

- 1. A Hopf action of $x \in H$ on $\mathbb{k}Q_0$*
- 2. A representation of $G(H)$ on $\mathbb{k}Q_1$ satisfying $s(g \cdot a) = g \cdot sa$ and $t(g \cdot a) = g \cdot ta$ for all $a \in Q_1$ and all $g \in G$.*
- 3. A $\mathbb{k}$ -linear endomorphism $\sigma : \mathbb{k}Q_0 \oplus \mathbb{k}Q_1 \rightarrow \mathbb{k}Q_0 \oplus \mathbb{k}Q_1$ satisfying*
 - 3.1 $\sigma(\mathbb{k}Q_0) = 0$*
 - 3.2 $\sigma(a) = e_{sa}\sigma(a)e_{h \cdot ta}$ for all $a \in Q_1$*

With this data, the x -action on $a \in Q_1$ is given by

$$x \cdot a = \gamma_{ta}a - \gamma_{h \cdot sa}(h \cdot a) + \sigma(a).$$

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 - 1.3.1 $\sigma(\mathbb{k}Q_0) = 0$;
 - 1.3.2 $\sigma(a) = e_{sa}\sigma(a)e_{g^{n-1}.ta}$ for all $a \in Q_1$;
 - 1.3.3 $\sigma(g \cdot a) = q^r g\sigma(a) - e_{g.sa}(1 - g^{n-1})(g \cdot a)e_{g^n.ta}$ for all $a \in Q_1$ and $g \in G$.

With this data, the x -action on $a \in Q_1$ is given by

$$x \cdot a = \gamma_{ta}a + (1 + q^r + q^{2r} + \cdots + q^{r(n-2)})\gamma_{sa}(g^{n-1} \cdot a) + \sigma(a).$$

2. *Every (filtered) action of R on $\mathbb{k}Q$ is of the form above.*

Context of Hopf Actions

Actions of
Hopf-Ore
Extensions on
Path Algebras

With a Hopf action, we have the following dictionary.

Language of Actions	Language of Categories
An H action on $T_{\mathbb{k}Q_0}(\mathbb{k}Q_1)$ is filtered if it preserves the path length grading.	$\mathbb{k}Q_0$ and $\mathbb{k}Q_1$ are H -stable.
A filtered H -action makes $\mathbb{k}Q_0$ into an algebra in $\text{rep}(H)$ and $\mathbb{k}Q_1$ into a $\mathbb{k}Q_0$ -bimodule in $\text{rep}(H)$.	$T_{\mathbb{k}Q_0}(\mathbb{k}Q_1)$ is a $\text{rep}(H)$ -tensor algebra.

$T_{\mathbb{k}Q_0}(\mathbb{k}Q_1)$ is a **minimal-faithful $\text{rep}(H)$ -tensor algebra** if $\mathbb{k}Q_1$ is an indecomposable $\mathbb{k}Q_0$ -bimodule in $\text{rep}(H)$ and no two-sided ideal of $\mathbb{k}Q_0$ in $\text{rep}(H)$ acts by 0 on $\mathbb{k}Q_1$.

Background

Actions of
Hopf-Ore Ext.

Applications

Future Directions