RESEARCH STATEMENT

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1. Overview

While focused in the field of algebra, specifically representation theory, my research traverses related areas of study using algebraic techniques. In particular, my dissertation research centers around studying representations and their properties in both the field of topological data analysis and the field of quantum symmetries. Topological data analysis (TDA), particularly persistent homology, is a young, dynamic field of mathematics that seeks to understand data by studying the features in the underlying "shape" of a discrete data set that persist as parameters change. This field has applications in a wide array of areas including biology, chemistry, economics, and machine learning. In applications, quantum symmetries refer to the features of spacetime particles that remain unchanged after some transformation. Motivated by observations in quantum mechanics, one of the mathematical lenses with which to study these phenomena is through Hopf algebras and their actions. By studying actions of Hopf algebras on path algebras of quivers, my work contributes to the larger program of classifying Hopf algebra actions on finite dimensional algebras over an algebraically closed field.

In each of these areas, I am prepared to continue my research with both faculty and student collaborators. For curious students, working together to further my research program can be an entry point to higher-level mathematics and the acquisition of valuable problem-solving skills. Students with little mathematical background are able to compute actions of Hopf algebras while students with some background in linear algebra would be able to engage with questions in the field of persistent homology. As a result, many students can begin participating in mathematical research early in their undergraduate career.

My passion for undergraduate research is driven by my own journey in the field of mathematics. As an undergraduate, I did not have the opportunity to participate in research, only discovering a love for it and the skills necessary to engage in it in graduate school. Because of this experience, I would like to create more opportunities for students from a wide range of backgrounds to engage in mathematics research and culture and develop problem solving skills earlier in their academic journeys. Whether it be through independent research projects, seminars and colloquia, or math club, I am motivated to introduce students to interesting mathematics.

This research statement outlines the work I have done in both topological data analysis and quantum symmetries before detailing specific ways students can be involved. In Section 2, I expound on topological data analysis and properties of the p-presentation distance between persistence modules. In Section 3, I provide more detail regarding my work in quantum symmetries by parameterizing the Hopf actions of a notable family of Hopf algebras. I conclude with ways students can become involved in mathematical research through my research program in Section 4.

2. Topological Data Analysis

Topological Data Analysis (TDA) is an exciting new field of interdisciplinary mathematics aimed at discovering underlying "shape" within discrete data sets [Car09]. Within this, my research is in the study of multiparameter persistence theory, which is closely related to quiver representation theory. This subfield focuses on understanding the "features" of data that persist as certain parameters are changed.

The TDA pipeline is typically represented as:

$$\text{Data} \longrightarrow \text{Geometry} \longrightarrow \text{Algebra} \longrightarrow \text{Summary} \longrightarrow \text{Analysis.}$$

As data is processed through, the result of the "Algebra" step is called a *persistence module*. This can be viewed as a representation of a quiver that encodes information about "features" of the data via vector spaces arising from homology theory. My research addresses questions of the stability of these objects under noisy data. This requires the notion of a metric, or distance, on algebraic objects, for which I utilize the p-presentation distance as defined by Bjerkevik and Lesnick in [BL21] and denoted as $d^p_{\mathcal{I}}$. My results provide a bound on the p-presentation distance between two finitely presented persistence modules.

Theorem 1. *Suppose M and N are finitely presented $\mathbb{R}^n$-persistence modules. For all $p \in [1, \infty]$*

$$d^{+p}_{\mathcal{I}}(M, N) \leq d^p_{\mathcal{I}}(M, N).$$

While the distance, $d^p_{\mathcal{I}}$, is often hard to calculate, I used this bound to compute the p-presentation distance between any two free indecomposable persistence modules. More specifically, for such persistence modules, generated at $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ respectively, $d^p_{\mathcal{I}}(M, N) = ||x - y||_p$. In the case that the free persistence modules have more than one generator, I have proven the following necessary and sufficient conditions for the p-presentation distance to be finite.

Theorem 2. *Let M and N be two finitely presented $\mathbb{R}^n$-persistence modules. Then the following are equivalent:*

i) $d^q_{\mathcal{I}}(M, N)$ is finite for some $q \in [1, \infty]$;
ii) $d^p_{\mathcal{I}}(M, N)$ is finite for all $p \in [1, \infty]$;
iii) there exists a pair of presentations of M and N that have the same underlying matrix;
iv) M and N have the same alternating sum of ungraded Betti numbers, i.e. $\sum_{i=1}^n (-1)^i \beta_i^M = \sum_{i=1}^n (-1)^i \beta_i^N$ where β_i^Z is the i^{th} ungraded Betti number of $Z \in \{M, N\}$.

2.1. Future Directions.

In addition to further investigation of metrics on persistence modules, I will begin to explore applications of this method of data analysis to draw conclusions about specific data sets. Doing so opens the door to advancing topological data visualization tools as well. Through the Applied Algebraic Topology Research Network (AATRN), I will be able to work with other collaborators to advance this field.

3. Quantum Symmetries

My work in this area extends the classical notion of symmetry, which can be encoded using the language of group actions, to the notion of quantum symmetry by using the language of Hopf actions. This is because Hopf algebras have a richer structure than group algebras which allow them to encode additional structures.

My work specifically studies Hopf actions on path algebras of quivers, which play a similarly fundamental role in noncommutative algebra to what polynomial rings play in commutative algebra. As such, my research extends work of Kinser and Oswald in [KO21] by classifying the actions of Hopf-Ore extensions of group algebras on path algebras of quivers. Hopf-Ore extensions of group algebras were classified by Panov in the following way.

Proposition 3. [Pan03] *Let G be a group and $R = \Bbbk G(x;\chi,\alpha)$ a Hopf-Ore extension of $\Bbbk G$. Then (up to change of variable) for a group character $\chi\colon G \to \Bbbk$ and for a linear form $\alpha : G \to G$ with $\alpha(uv) = \alpha(u) + \chi(u)\alpha(v)$ for all $u, v \in G$ we have:*

(i) $\Delta(x) = 1 \otimes x + x \otimes h$ for some h in the center of G,
(ii) the relation $xg = \chi(g)gx + \alpha(g)(1 - h)g$ holds for all $g \in G$.

Utilizing Panov's results enabled me to parametrize the actions of Hopf-Ore extensions of group algebras on path algebras. The following theorem statement is an abbreviated, version of the full result in [AK].

Theorem 4. *Let Q be a quiver. The following data determines a (filtered) Hopf action of R as classified above, on $\Bbbk Q$, and all such actions are of this form.*

(1) A Hopf action of R on $\Bbbk Q_0$ determined by a permutation action of G on Q_0 and a collection of scalars satisfying a relation determined by (ii);
(2) A representation of G on $\Bbbk Q_1$ compatible with the $\Bbbk Q_0$-bimodule structure on $\Bbbk Q_1$;
(3) A $\Bbbk$-linear endomorphism of $\Bbbk Q_0 \oplus \Bbbk Q_1$ satisfying some technical conditions explicitly described in [AK].

I then apply this general result to some specific situations, including several Noetherian Hopf algebras of Gelfand-Kirillov dimension one and two, as detailed in [AK].

3.1. Future Directions.

My work provokes curiosities regarding actions of more general Hopf-Ore extensions on path algebras. On one hand, I will extend my work by replacing group algebras with more general classes of Hopf algebras. On the other hand, I will classify the action of the element $x \in T$ when Hopf-Ore extensions are defined by allowing x to have comultiplication of the form $\Delta(x) = x\otimes a + b\otimes x + v(x\otimes x) + w$ for $a, b \in A$ and $v, w \in A\otimes A$. This comultiplication is motivated by Hopf algebras such as the coordinate algebra of the Heisenberg group of dimension three, which is not a Hopf-Ore extension since the adjoined element is not skew-primitive. Another well known Hopf algebra, $U_q(sl_2(\Bbbk))$, cannot be written as a Hopf-Ore extension of a group algebra as it is generated by two skew-primitive elements instead of one. This invokes the expansion of the notion of an iterated Hopf-Ore extension of $\Bbbk$ as defined in [BOZZ15, Definition 3.1] to that of an iterated Hopf-Ore extension of the group algebra $\Bbbk G$ in the following way.

Definition 5. *Let G be a finitely generated abelian group and $\Bbbk G$ be its group algebra possessing the standard Hopf algebra structure. An* ***iterated Hopf-Ore extension*** *of $\Bbbk G$ is a* *sequence of Hopf* $\Bbbk$*-algebras*

$$H_0 := \Bbbk G \subset H_1 \subset H_2 \subset \cdots \subset H_n$$

where at each successive step, $H_i = H_{i-1}[x_i; \tau_i, \delta_i]$ *is a Hopf-Ore extension of* H_{i-1} *and there exists* $a_i, b_i \in \Bbbk G$*, and* $v_i, w_i \in \Bbbk G \otimes \Bbbk G$ *such that for all* $1 \leq i \leq n$*,* $\Delta(x_i) = x_i \otimes a_i + b_i \otimes x_i + v_i(x_i \otimes x_i) + w_i$*.*

In addition to classifying the Hopf actions of these algebras on path algebras, I will determine their GK-dimensions and Krull-dimensions, investigate a conjecture about the forms of v_i and w_i in the expression for $\Delta(x_i)$ above, and determine when iterated Hopf-Ore extensions have other properties such as being Noetherian or AS-regular.

4. Strategies for the Integration of Scholarship and Teaching

I am driven to include students in my scholarship so they develop problem-solving skills and have access to research experiences. So this can happen, I have designed thoughtful ways for students with very little mathematical background to get involved. While knowledge of linear algebra is preferred for students to engage in research activities on the theoretical side of TDA, even less is needed for students who would like to explore topological data visualization or quantum symmetries. Once students choose an area they are interested in, they will quickly be engaged in original work.

Unlike other, more classical areas of mathematics, Topological Data Analysis serves as a great entry point for undergraduates who are curious about advanced mathematics. As a first introduction to this field, I designed a course using backwards design through the Center for the Integration of Research, Teaching, and Learning (CIRTL)'s Turning Your Research into Teaching workshop. This course is accessible to students with an array of mathematical backgrounds as it relies on the visual nature of TDA to help students engage in an intuition and example driven manner. Furthermore, TDA also offers the possibility of using software such as RStudio and visualization tools, thus appealing to students from a variety of majors. Thus TDA bridges the gap between pure mathematics, applied mathematics, and computer science.

Students interested in abstract algebra or who enjoy calculations, would be able to contribute to the field of quantum symmetries quickly. First, I will equip students with the necessary background and notation needed. Then they will parameterize the actions of a certain Hopf algebra on path algebras of quivers. For students who would like to learn more theory, I would guide them through more advanced material relating to this area.

Amidst the excitement of research, students will gain other valuable skills. For instance, I will teach students about helpful resources and how to best use them. I will coach students on how to shape their observations into well-posed mathematical questions through a hands-on approach. Finally, students will learn to effectively communicate mathematics through collaboration with myself and other students, writings, and by presenting at seminars and conferences. I am passionate about inviting students into the exciting world of mathematical research using a holistic approach that provides them experiences, opportunities, and skills I would have benefited from during my undergraduate career.

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